Optimization Theory and Methods

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- Transit Ridership Example
- What is Linear Optimization?
- Set Notation Review
- Linearizing Non-linear Problems

↳ Example: Transit Ridership

- A transit agency is performing a review of its services. It has decided to measure its overall effectiveness in terms of the total number of riders it serves. The agency operates a number of modes of transport. The table shows the average number of riders generated by each trip (by mode) and the cost of each trip (by mode)

Mode	Heavy Rail	Light Rail	BRT	Bus
Avg. ridership per trip (r_i)	400	125	60	40
Avg. cost per trip (c_i)	200	80	40	30

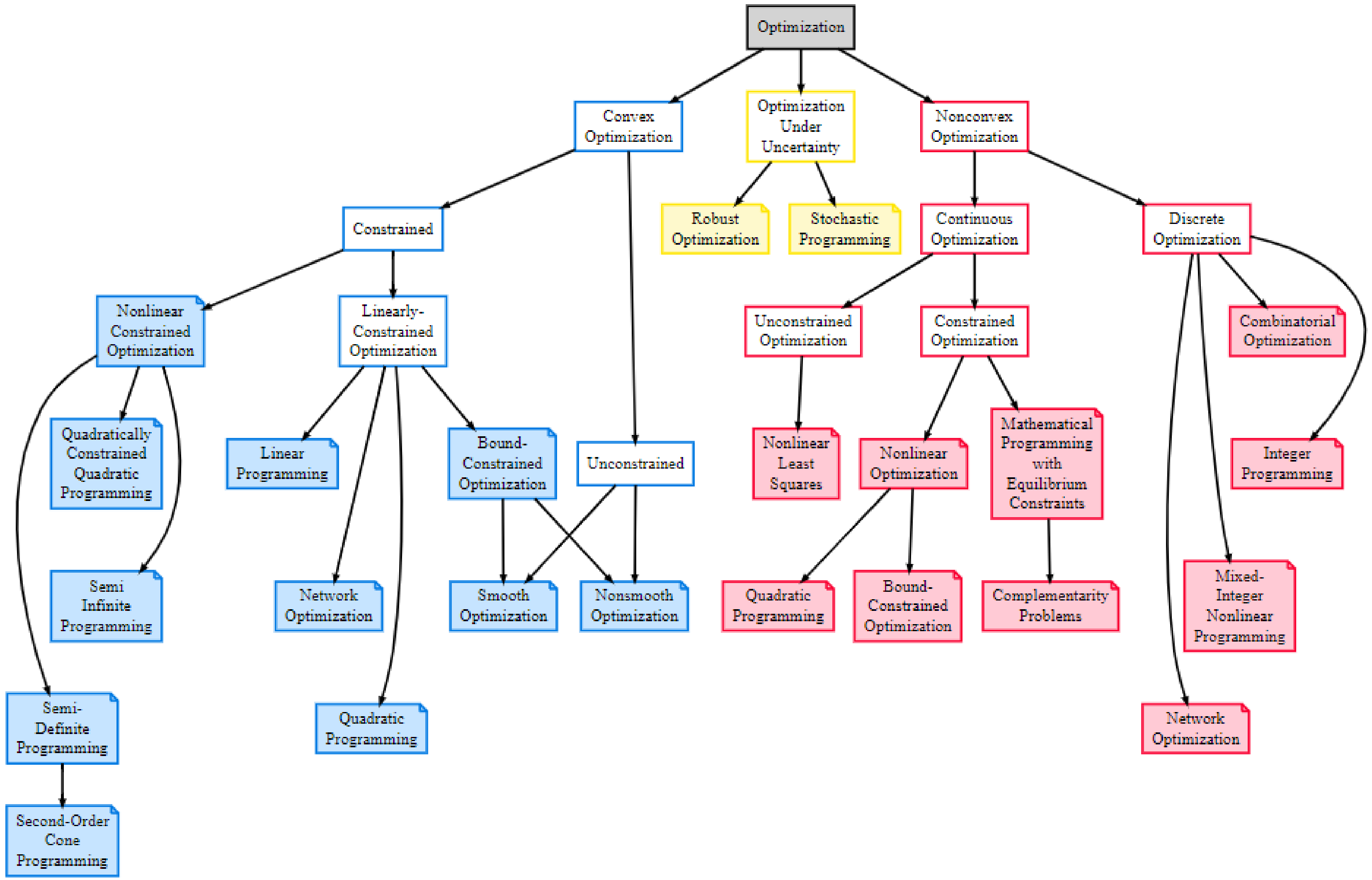
- Give a formulation of the problem to maximize the total number of riders the agency services given a fixed daily budget of $5,000.

↳ The Main Components of an Optimization Problem

- **Objective function**
  - Summarizes the ***objective*** of the problem (MAX or MIN)
- **Constraints**
  - ***Limitations*** placed on the problem: control allowable solutions.
  - Problem ***statement***: ‘given….’, ‘must ensure….’, ‘subject to….’
  - ***Equations***, or inequalities, or variable value types.
- **Decision variables**
  - Quantities or decisions to be determined.
  - Multiple types (real numbers, non-negative reals, integers, binary).

An optimization problem with linear objective function AND linear constraints is called a linear program/a linear *optimization problem*.

↳ Types of Optimization Problems



- There are multiple ways to ***develop a model formulation***. One way is as follows:

1) **Decide on an initial set of decision variables**
   - Traditionally letters from the end of the alphabets; use of subscripts; ordering of subscripts.
2) **Determine objective function**
   - Obtainable from problem statement
   - Can be very complex
3) **Determine the constraints**
   - Variable-value constraints: non-negativity, binary constraints.
   - Capacity constraints, demand constraints, flow balance constraints.
   - Sometimes necessitates introduction of additional variables.

↳ Set Notation Review

- ***Set***: collection of distinct objects
- $\mathbb{R}$: set of real numbers
- $\mathbb{Z}$: set of integers
- Superscript $+$: non-negative elements of a set
- $\in$: 'is an element of'
- $\{\quad\}$: 'the set containing' (members of the set are placed between curly brackets)
- : or | : 'such that' example: $\{x \in S: x \geq 0\}$
- $\exists$: 'there exists'
- $\forall$: 'for all'

↳ Back to Transit Ridership Example

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Decision variables?

- x_1 = Number of trips made using heavy rail
- x_2 = Number of trips made using light rail
- x_3 = Number of trips made using bus rapid transit (BRT)
- x_4 = Number of trips made using bus

Objective function?

- MAX (Total Ridership)
- Ridership $= 400 * x_1 + 125 * x_2 + 60 * x_3 + 40 * x_4$

Constraints?

- Cost budget = $5,000
- Cost $= 200 * x_1 + 80 * x_2 + 40 * x_3 + 30 * x_4$

- **MAX** $(400 * x_1 + 125 * x_2 + 60 * x_3 + 40 * x_4)$
- **s.t.**

$$200 * x_1 + 80 * x_2 + 40 * x_3 + 30 * x_4 \leq 5000$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- **Generalization:**

- M = set of modes
- r_i = average ridership per trip for mode i
- c_i = average cost per trip for mode i

MAX $(\sum_{i \in M} r_i x_i)$

s.t.

$$\sum_{i \in M} c_i x_i \leq 5000$$

$$x_i \in \mathbb{R}^+, \forall i \in M$$

↳ Transit Ridership: Additional Constraints

The agency wants to provide a minimum number of trips $\boldsymbol{m_i}$, for each mode $\boldsymbol{i}$

$$x_1 \geq m_1$$
$$x_2 \geq m_2$$
$$x_3 \geq m_3$$
$$x_4 \geq m_4$$

Generalization →

$$x_i \geq m_i, \forall i \in M$$

The agency wants to provide service to a minimum number of riders $\boldsymbol{b_i}$, for each mode $\boldsymbol{i}$

$$r_1 x_1 \geq b_1$$
$$r_2 x_2 \geq b_2$$
$$r_3 x_3 \geq b_3$$
$$r_4 x_4 \geq b_4$$

Generalization →

$$r_i x_i \geq b_i, \forall i \in M$$

- In a Linear Programming problem (LP),
  - Objective function AND constraints MUST BE linear
  - Variable type must be continuous
    - ✓E.g. real numbers, non-negatives, are OK
    - ✓Integers, binary are NOT OK

- $MAX\{x_1, x_2, \dots\}$, $x_i * y_i$, $|x_i|$, etc. => non-linear if x_i and y_i are variables
  - Sometimes there is a way to convert these types of constraints into linear constraints by adding some decision variables.
  - The key is to ensure that the transformation holds at the optimal solution.
  - Sometimes you need to expand the set of feasible solutions, which is ok.

↳ Dealing with Absolute Values

Example: $MIN\ 5X + 2|Y|$
s.t.
$X + Y \geq 9$

=> **OPTION 1:**

$|Y| = MAX\{Y, -Y\} = V$

Replace by $V \geq Y$ and $V \geq -Y$

Formulation1:

$MIN\ 5X + 2V$
s.t.
$V \geq Y$
$V \geq -Y$
$X + Y \geq 9$

=> OPTION 2:

Introduce new variables Y^+, Y^- such that:

$Y^+, Y^- \geq 0$ and $Y = Y^+ - Y^-$

We want $Y = Y^+$ or $Y = -Y^-$, depending on sign of Y.

Then, $Y = Y^+ - Y^-$ and $|Y| = Y^+ + Y^-$

Formulation 2:

$MIN\ 5X + 2Y^+ + 2Y^-$
s.t.
$X + Y^+ - Y^- \geq 9$
$Y^+, Y^- \geq 0$

Why are these two formulations valid?

↳ Why is Formulation 1 Valid?

- $V = |Y| = MAX\{Y, -Y\}$ is OK: It's just a variable change.
- But, $V \geq Y$ and $V \geq -Y$ means we are replacing $V = MAX\{Y, -Y\}$ by $V \geq MAX\{Y, -Y\}$.
- Is this a problem? In other words, does there exist a possibility that we may get an optimal solution where $V > MAX\{Y, -Y\}$?
- The answer is NO!! Let's see why.
- Let there be an optimal solution with $V = MAX\{Y, -Y\} + \Delta$.
- Then we can reduce the value of V by Δ, and reduce the objective function value WITHOUT VIOLATING ANY OF THE CONSTRAINTS in formulation 1.
- …which means that such a solution cannot be optimal.
- So formulation 1 is valid!

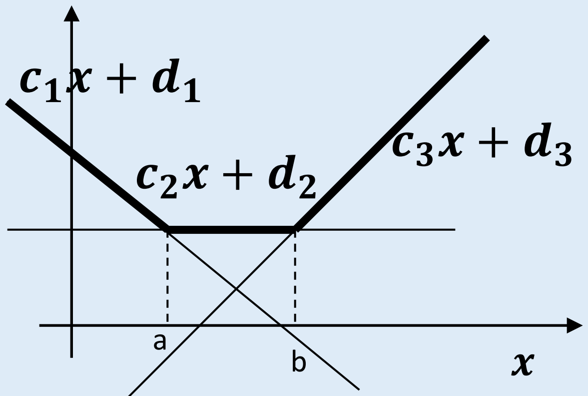
↳ Why is Formulation 2 Valid?

- $Y = Y^+ - Y^-$ and $|Y| = Y^+ + Y^-$: This is OK as long as at most one of Y^+ and Y^- is non-zero.
- Does there exist a possibility of an optimal solution where,

  $Y^+ > 0$ AND $Y^- > 0$?
- The answer is NO!! Let's see why.
  - Let there be such an optimal solution with $Y^+ > 0$ AND $Y^- > 0$.
  - Let's say $Y^+ \geq Y^-$.
  - Then we can reduce both Y^+ and Y^- by an amount equal to Y^- and reduce the objective function value further (below the supposedly optimal solution) without violating any constraints.
- ...which means that such a solution cannot be optimal.
- So formulation 2 is valid!

Example: Objective function defined as

$$c(x) = \begin{cases} c_1 x + d_1, & \forall x \in (-\infty, a) \\ c_2 x + d_2, & \forall x \in [a, b] \\ c_3 x + d_3, & \forall x \in (b, +\infty) \end{cases}$$



Introduce a new variable T with objective function MIN T s.t. $T = \max(c_1 x + d_1, c_2 x + d_2, c_3 x + d_3)$

$$\begin{aligned} &\min T \\ &\text{s.t.} \quad T \geq c_1 x + d_1 \\ &\qquad\;\; T \geq c_2 x + d_2 \\ &\qquad\;\; T \geq c_3 x + d_3 \end{aligned}$$

Exercise: Construct an argument for the validity of this formulation.

Objective :

Key Concepts：